**1** Consider a particle of mass $m$ in a one-dimensional infinite square well between $x = 0$ and $x = a$. The normalized energy eigenstates are $\psi_n(x)$ with corresponding energies $E_n$, where $n = 1$ is the ground state. The particle starts in a state given by, [5, 5, 10, 5]

$$\Psi(x, t = 0) = \sqrt{\frac{4}{5}}\,\psi_1(x) - \sqrt{\frac{1}{5}}\,\psi_2(x)$$

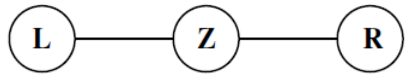
a) If you measure the energy of the particle, what values will you get and with what probability?

b) Now you make a position measurement of the particle. Are you more likely to find the particle in the left half of the well ($x < a/2$), or the right half of the well ($x > a/2$)? Justify your answer.

c) Assuming the particle is in the state $\Psi(x, t = 0)$ given above, suppose you make a high precision position measurement of the particle. After this position measurement, you immediately measure the energy of the particle. At this point, what possible value(s) could you get for the energy of the particle? (No need to find the probabilities of each.) Clearly explain how you arrive at your answer.

d) Consider the same scenario as in c), but now instead of making an immediate energy measurement, consider waiting some time. Would the probability of measuring an energy of $E_2$ depend on how long you wait between the position measurement and the energy measurement? Explain how you arrive at your answer.

**2**. Consider an electron in a linear molecule consisting of three atoms, where the central atom *Z* is located between the left and right atoms *L* and *R* like in the figure below.




$|L\rangle$, $|Z\rangle$, and $|R\rangle$ denote three orthonormal vectors that correspond to the electron localized at the atoms *L*, *Z*, and *R*, respectively, and span the three-dimensional Hilbert space. With respect to the basis $\{|L\rangle, |Z\rangle, |R\rangle\}$ the Hamiltonian of the electron has the matrix form

$$\mathcal{H} = \begin{pmatrix} b & -a & 0 \\ -a & b & -a \\ 0 & -a & b \end{pmatrix}, \text{ with } a > 0. \qquad [5, 5, 5, 10]$$

a) Verify that the three states $|0\rangle$, $|+\rangle$, and $|-\rangle$ given by,

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle); \qquad |\pm\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}\big(|L\rangle \mp \sqrt{2}|Z\rangle + |R\rangle\big)$$

are eigenstates of $\mathcal{H}$ and calculate their corresponding eigenenergies. Show your work!

b) Let the electron be in the state $|-\rangle$. Calculate the probability to find the electron in state $|Z\rangle$ (i.e. localized on atom *Z*).

c) Calculate the expectation value $\langle\mathcal{H}\rangle$ and the variance $\sigma_{\mathcal{H}}^2$ of the Hamiltonian in the state $|L\rangle$.

d) At $t = 0$ the electron is localized on atom Z, i.e. it is in the state $|\Psi(0)\rangle = |Z\rangle$. Calculate the time evolution of the state $|\Psi(t)\rangle$ and the probability $P(t) = |\langle Z|\Psi(t)\rangle|^2$ to find the electron in the state $|Z\rangle$ for $t > 0$.

**3**. All of the questions below pertain to a general quantum state $|S(t)\rangle$ in Hilbert space.

[5, 10, 10]

(a) Using the completeness property of the momentum eigenstates $|p\rangle$ express the general quantum state $|S(t)\rangle$ in the momentum basis.

(b) Derive the transformation from the momentum-space wavefunction $\Phi(p,t)$ to the "energy space" wavefunction $c_n(t) = \langle n|S(t)\rangle$, where $\widehat{\mathcal{H}}|n\rangle = E_n|n\rangle$, with $n = 1,2,3, \ldots$.

(c) Find the momentum operator in the basis of simple harmonic oscillator energy eigenstates. In other words, express $\langle n|\hat{p}|S(t)\rangle$ in terms of $c_n(t) = \langle n|S(t)\rangle$. {*Hint: the matrix elements of the momentum operator in the harmonic oscillator basis are given by:* $\langle n|\hat{p}|n'\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n}\,\delta_{n',n-1} - \sqrt{n'}\,\delta_{n,n'-1}\right)$}

**4**. Consider a particle of mass $m$ subjected to a *finite* spherical well potential:

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r > a \end{cases} \quad \text{with } V_0 > 0.$$

We are interested in *bound states* of this potential. [5, 5, 10, 5]

(a) Make a plot of the potential $V(r)$ as a function of $r$. What is the range of possible bound state energies $E$ for this potential (give both the lower and upper limits)?

(b) Since this potential is spherically symmetric, the time-independent Schrodinger equation in spherical coordinates simplifies considerably, and the solution is of the form $\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi)$. Write out the radial equation for $u(r) \equiv rR(r)$ that results for this problem. (Don't derive it, just copy it from the formula sheet!)

(c) Now assume that $\ell = 0$ and solve for $u(r)$ in the regions $r \leq a$ and $r > a$, making sure that both solutions for $u(r)$ (and $R(r)$) are finite in their respective domains.

(d) Enforce the continuity conditions for the two solutions at $r = a$ and derive an implicit (transcendental) equation for the bound state energies $E$. Do not try to solve for $E$, unless you are really bored!